# *A simple proof of the rationality of the anharmonic ratio of four faces of a zone.*

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(Communicated by Prof. H. A. MIERS.)

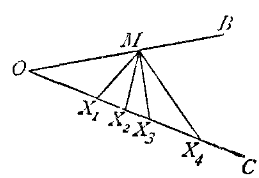
[Read November 13th, 1900.]

THE anharmonic ratio here considered is that of the pencil ($P$) formed by the four lines (lying in a plane perpendicular to the zonal axis) drawn from any point $M$ perpendicular to four faces of the zone. The anharmonic ratio of the pencil $P$ is evidently independent of the position of $M$.

Let four planes be drawn through $M$ perpendicular to the four lines of the pencil $P$; these are parallel to the crystal faces and meet in a line (the line through $M$ perpendicular to the plane of the pencil $P$ and parallel to the zonal axis).

These four planes are cut by the plane of the pencil in a pencil having the same angles, and therefore the same anharmonic ratio, as the pencil $P$. Now it is a well-known theorem that the pencil in which four planes through a line are cut by *any* plane, has a constant anharmonic ratio; hence the anharmonic ratio of the pencil $P$ is equal to the anharmonic ratio of the pencil in which the four planes through $M$ parallel to the crystal faces are cut by any plane ($\pi$).

Let $OA$, $OB$, $OC$, be the positions of the crystallographic axes, and $a$, $b$, $c$ their lengths. We noticed above that the anharmonic ratio of the pencil $P$ is independent of the position of its vertex $M$; we may take $M$ therefore on the axis $OB$, and take the plane of $OB$, $OC$ as the plane ($\pi$).




Let $h_1k_1l_1$, $h_2k_2l_2$, $h_3k_3l_3$, $h_4k_4l_4$ be the indices of the four faces; then their intercepts on the axes are

$$\frac{a}{h_1}, \frac{b}{k_1}, \frac{c}{l_1}, \text{ \&c.}$$

Let $OC$ cut the four planes through $M$ parallel to the crystal faces in $X_1$, $X_2$, $X_3$, $X_4$. Then we have proved that the pencil $P$ has the same anharmonic ratio as the pencil $M\ (X_1X_2X_3X_4)$; *i.e.* as the range $(X_1X_2X_3X_4)$.

Therefore the anharmonic ratio of the pencil $P$

$$= \frac{(OX_3 - OX_1)\ (OX_4 - OX_2)}{(OX_4 - OX_1)\ (OX_3 - OX_2)}.$$

Now, $$\frac{OX_1}{OM} = \frac{\frac{c}{l_1}}{\frac{b}{k_1}} = \frac{ck_1}{bl_1}, \text{ \&c.}$$

or $$OX_1 = OM\,\frac{ck_1}{bl_1};\quad OX_2 = OM\,\frac{ck_2}{bl_2};\quad OX_3 = OM\,\frac{ck_3}{bl_3};\quad OX_4 = OM\,\frac{ck_4}{bl_4}$$

and therefore the ratio of the pencil $P$

$$= \frac{\left(\frac{ck_3}{bl_3} - \frac{ck_1}{bl_1}\right)\left(\frac{ck_4}{bl_4} - \frac{ck_2}{bl_2}\right)}{\left(\frac{ck_4}{bl_4} - \frac{ck_1}{bl_1}\right)\left(\frac{ck_3}{bl_3} - \frac{ck_2}{bl_2}\right)} = \frac{(l_1k_3 - l_3k_1)\ (l_2k_4 - l_4k_2)}{(l_1k_4 - l_4k_1)\ (l_2k_3 - l_3k_2)},$$

which is rational, for $l_1$, $l_2$, $l_3$, $l_4$ and $k_1$, $k_2$, $k_3$, $k_4$ are rational.

Exactly similarly it may be proved that the ratio $P$

$$= \frac{(k_1h_3 - k_3h_1)\ (k_2h_4 - k_4h_2)}{(k_1h_4 - k_4h_1)\ (k_2h_3 - k_3h_2)} = \frac{(h_1l_3 - h_3l_1)\ (h_2l_4 - h_4l_2)}{(h_1l_4 - h_4l_1)\ (h_2l_3 - h_3l_2)}.$$