A new stereographic protractor.[1]

(*With Plate IV.*)

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[Read March 11, 1913.]

IN the stereographic projection (fig. 1) $ABA'B'$ is the primitive circle, having the same radius a as the corresponding sphere; $ATA'T'$ is the circle representing a great circle on the sphere inclined at an angle θ to the plane of the projection; O and C are the centres of the primitive circle and the circle $ATA'T'$ respectively. CO are joined and projected both ways to meet the circles as shown. Then we have the following simple relations:—

$$CO = a \cot\theta, \quad OT = a \tan\frac{\theta}{2}, \quad OT' = a \cot\frac{\theta}{2}, \quad \text{and} \quad CA = a \operatorname{cosec}\theta.$$

By means, therefore, of a scale,[2] which is graduated on the one side of the zero according to the tangent of the angle and on the other according to the tangent of half the complement of the angle, the points T and C are at once read off, and the corresponding circle is readily drawn so long as its radius is not too large.

When $\theta = 45^\circ$, $OC = a$; C then lies on the primitive circle, and coincides with B. As the azimuthal angle grows larger the corresponding radius rapidly increases in length, and recourse must be had to a beam-compass, and ultimately to a flexible ruler. Unless the scale of the stereogram is small, the centres of the circular arcs passing near the pole of the projection lie at an inconvenient distance outside the primitive circle, and it is probably for that reason that Penfield[3] selected as radius

[1] Communicated by permission of the Trustees of the British Museum.

[2] Cf. A. Hutchinson, 'On a protractor for use in constructing stereographic and gnomonic projections of the sphere.' Min. Mag., 1908, vol. xv, pp. 93–112.

[3] S. L. Penfield, Amer. Journ. Sci., 1902, ser. 4, vol. xiv, p. 251.

of the primitive circle 7 cm., while Hutchinson[1] recommended much the same size, viz. $2\frac{1}{2}$ inches (6·35 cm.). The author, on the other hand, is of opinion that this scale, while no doubt admirably suited to teaching purposes, is too small for the convenient discussion of complex crystals, and prefers to follow Fedorov[2] and use a primitive circle of radius 10 cm. Its size is not inconvenient so long as none of the drawing operations need be conducted far outside it; but to achieve this object, some means other than the ordinary method of compasses must be found for constructing circles corresponding to great circles of azimuthal angle ranging from 45° to 90° measured from the plane of the projection.

The curved ruler devised by Fedorov and Wulff[3] for drawing flat circular arcs consists very simply of a stiff spring, which is bent by

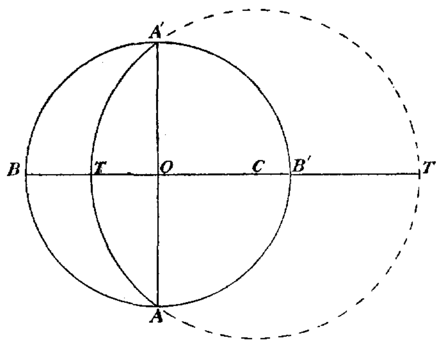


FIG. 1.

a wide two-pronged fork arrangement pushing it against two fixed pins, and returns to its original flat shape by its flexural elasticity, the distance between the prongs being about three-quarters or so that between the pins. The apparatus was afterwards modified[4] by the addition of a scale, which was graduated to give the azimuthal angle to which the particular arc corresponded, the range extending from 0° to 25°, measured from the pole of the projection.

[1] A. Hutchinson, loc. cit., p. 94.

[2] E. S. Fedorov, Zeits. Kryst. Min., 1893, vol. xxi, p. 622.

[3] E. S. Fedorov, Zeits. Kryst. Min., 1893, vol. xxi, p. 618. G. Wulff, ibid., p. 253. Vide also S. L. Penfield, Amer. Journ. Sci., 1901, ser. 4, vol. xi, p. 138.

[4] E. S. Fedorov, Zeits. Kryst. Min., 1902, vol. xxxvii, p. 142.

It will be noticed that this device does not fill the whole range from 45° to 90°; moreover, the curve for a particular reading of the scale differs sensibly according as the spring is being pushed or is returning by its own elasticity, and the curves of greater curvature are not even approximately circular. The curve assumed by a bent spring was first studied by Bernoulli,[1] who worked out its principal properties, and gave it the name *elastica*. The first approximation to it is the harmonic curve or curve of sines, and the curvature is constant only for very small amounts of bending. The curve of the flexible ruler described above is really composed of three separate elasticas, the outer two of which are similar; the curvature is by this means maintained more uniform than if the spring were bent by a central push in the way that an archer bends his bow.

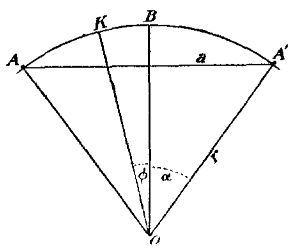


FIG. 2.

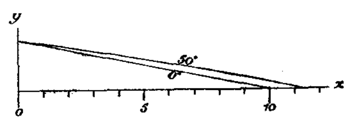


FIG. 3.

The problem considered by the author was the practicability of obtaining more regular curvature, even for considerable amounts of bending the spring, by varying its flexural rigidity, or in other words its thickness.

Fig. 2 shows a spring bent into the form of a circle of radius r by a central push at B against two fixed pins A and A', distant $2a$ apart. K is any point on the spring between A and B. O, the centre of the circle, is joined to the points A, A', B, K. The angles AOB, KOB are α and ϕ respectively; the thickness of the spring at K is y, and KB, the distance of K from the centre of the spring measured along it, is x.

Then $x = r\,\phi$, and $r = a$ cosec α.

By taking moments about K we obtain the equation

$$yc = Qr^2 \sin(\alpha - \phi), \text{ or } Qa^2 \frac{\sin(\alpha - \phi)}{\sin^2 \alpha},$$

[1] J. Bernoulli, 'Sur la force et la courbure des lames élastiques.' Mém. Acad. Berlin, 1766.

Q being the pressure exerted at the pins A and A', and finally we have

$$y = y_0 \frac{\sin\left(a - \frac{x}{a} \cdot \sin a\right)}{\sin a},$$

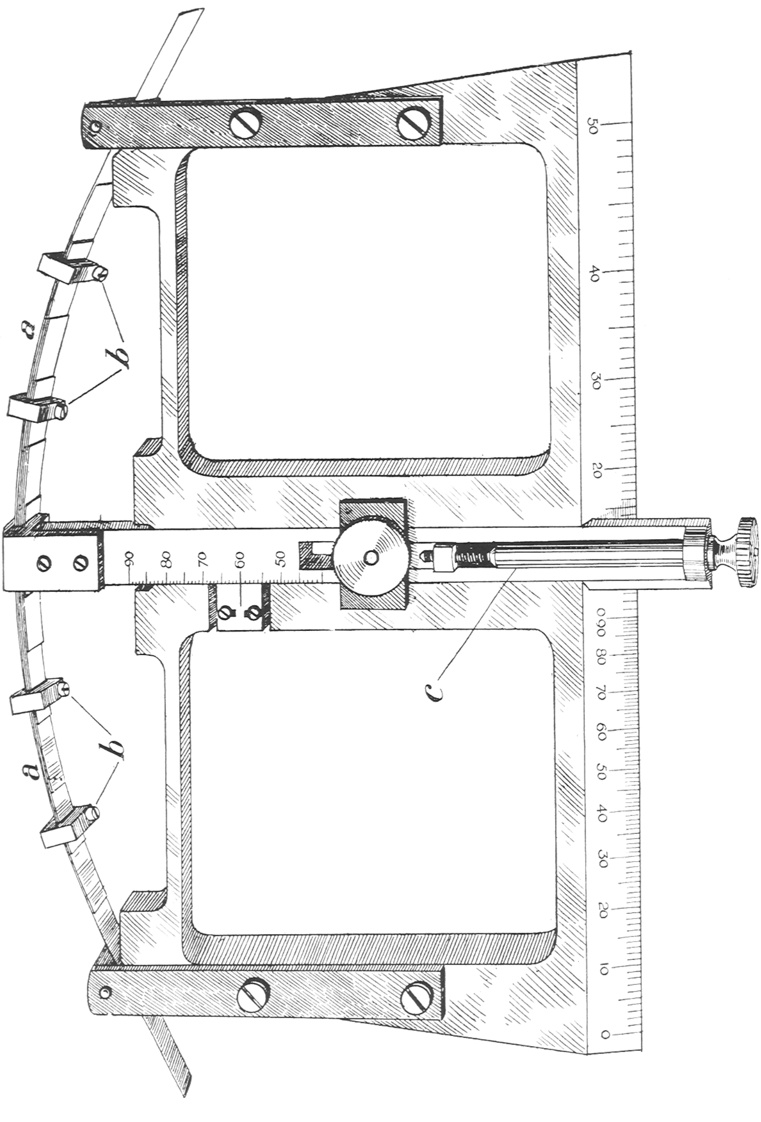
y_0 being the thickness at the middle point B.

We see that, since the function on the right-hand side is not independent of a, the law of thickness for uniform curvature is not constant for different curvatures. Fig. 3 illustrates the extent of this variability, the curves representing the relative thickness for the values 0° and 50° of a. The angle a, it may be noted, is the azimuth of the corresponding great circle measured from the pole of the projection. Since the curves lie close together, an approximate solution of the problem, sufficiently accurate for graphical purposes, is obtainable by using a spring corresponding in shape intermediately to the curves depicted.

In the instrument as designed (Plate IV) a multiple spring (a) composed of a series of clock springs of suitable lengths, pinned together, is used instead of a single spring of varying thickness, because the latter would have been so much less readily obtainable, and the result is much the same. Screw-grips (b) are employed in place of rivets, because the holes required for the latter seriously disturb the rigidity of the springs at those points; they are not screwed too tightly, because a little play is necessary to prevent kinking in the outer spring giving the actual trace required; and they are cut away in front so as not to interfere with the passage of the pen or pencil. The pushing-piece (c) is provided with a clamp and fine adjustment of the usual pattern, and carries a scale giving the azimuth of the corresponding great circle, the range extending from 40° to 90°. In using the scale, it is, of course, necessary to allow for the thickness of the pen or pencil used. The fixed pins against which the spring impinges are square in section, the distance between the inner corners being 20 cm. On the opposite side of the frame the bevelled edge carries two scales, of the kind mentioned above, viz. one of tangents of half the complement of the angle, from 0° to 90°, giving the position of T (fig. 1), and the other of tangents of the angle, from 0° to 50°, giving the position of C, the unit being 10 cm. and the scales being graduated to degrees. Since the range of the spring-ruler extends from 40° to 90°, the two methods overlap between 40° and 50°. With this stereograph the sheet of paper used need not be much larger than the stereogram itself.

It may be remarked that theoretically better results would be given by a double pusher after the pattern used by Fedorov and Wulff, but the mechanical construction would have been immensely complicated owing to the necessity of maintaining the centre of the spring in the line at right angles to that joining the two pins. Moreover, it is advantageous to have a rigid connexion between the spring and the pushing piece.

The frame is constructed of gun-metal, except for the fixed pins, which are of steel, and the scales are silvered. The instrument was made for the Mineral Department of the British Museum by Messrs. J. H. Steward, Limited, 457, West Strand, London, W.C.



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