# *On the determination of the optic axes of a crystal from extinction-angles.*

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§ 1. VARIOUS authors have discussed the problem how to determine the position of the optic axes of a biaxial crystal when the extinction-angles on $n$ different faces have been observed. It has been proved that a unique solution is possible when $n = 1$, 2, or 4, according as the crystal is orthorhombic, monoclinic, or triclinic. It is assumed (and will be assumed in this paper) that the $n$ faces have general positions. The result is not true, if one or more of the faces has a specialized position. For instance, the extinction-angle on a face of an orthorhombic crystal which is parallel to an axis of symmetry will obviously not suffice to determine the position of the optic axes.

Moreover, graphical methods have been given to determine the positions of the optic axes from measurements of the extinction-angles. These graphical methods depend for the most part on processes of trial and error, using the stereographic net or some such device; while the theoretical investigations involve the use of rather complicated formulae.[1] I propose in this paper to reconsider the problem, basing the investigation on well-known geometrical results, and giving those graphical constructions which require only the theoretically simplest apparatus for their performance.

§ 2. Suppose the poles of the crystal-faces are drawn on a sphere with centre $O$ and unit radius. Let $A$, $B$, $C$, $D$ be the poles of four crystal-faces; and let $aa'$, $bb'$, $cc'$, $dd'$ be the perpendicular pairs of great circles through $A$, $B$, $C$, $D$ respectively which touch the extinction-directions on the faces with poles $A$, $B$, $C$, $D$. In the case of a triclinic crystal the result to be proved is: Given $aa'$, $bb'$, $cc'$, $dd'$ in any positions (not too specialized), there is one and only one pair of real points $H$, $K$ on the

[1] L. Weber, Zeits. Krist., 1921, vol. 56, pp. 1–11, 96–103; A. Johnsen, Centralblatt Min., 1919, pp. 321–325 [Min. Abstr., vol. 1, p. 222].

sphere, such that the great circles $HA$ and $KA$ are harmonic conjugates for $a$ and $a'$; and so for $B$, $C$, $D$. For the fundamental property of the ends $H$, $K$ of the optic axes through $O$ is that $aa'$ are the bisectors of the angles between $HA$ and $KA$.

Take $O$ as origin of rectangular Cartesian co-ordinates. Let

$$\mathbf{A}\lambda^2+\mathbf{B}\mu^2+\mathbf{C}\nu^2+2\mathbf{F}\mu\nu+2\mathbf{G}\nu\lambda+2\mathbf{H}\lambda\mu=0 \qquad \text{(i)}$$

be the tangential equation of a cone of the second degree with vertex $O$ (the condition that it should touch the plane $\lambda x+\mu y+\nu z=0$). If the planes of $a$ and $a'$, namely

$$l_1x+m_1y+n_1z=0,\ l_1'x+m_1'y+n_1'z=0 \qquad \text{(ii)}$$

are conjugate with respect to the cone (i),

$$\mathbf{A}l_1l_1'+\mathbf{B}m_1m_1'+\mathbf{C}n_1n_1'+\mathbf{F}(m_1n_1'+m_1'n_1)+\mathbf{G}(n_1l_1'+n_1'l_1) +\mathbf{H}(m_1n_1'+m_1'n_1)=0. \qquad \text{(iii)}$$

If the planes of each pair $aa'$, $bb'$, $cc'$, $dd'$ are conjugate with respect to the cone, we have four linear relations such as (iii) connecting $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$; and therefore the cones of the second degree with vertex $O$ for which the planes of $aa'$, $bb'$, $cc'$, $dd'$ are all conjugate form a tangential pencil; i.e. they all touch four fixed planes. Now the planes of each pair $aa'$, $bb'$, $cc'$, $dd'$ are conjugate with respect to $OH$ and $OK$ considered as together forming a degenerate cone with vertex $O$. Hence $OH$ is the intersection of one pair of the four fixed planes and $OK$ is the intersection of the other pair. Also the cone with tangential equation

$$\lambda^2+\mu^2+\nu^2=0$$

belongs to the tangential pencil, for the planes of $a$ and $a'$, etc., are perpendicular. Hence the cones of the tangential pencil are confocal, and $OH$, $OK$ are a pair of focal lines. But it is well known that a confocal family of cones has three pairs of focal lines lying one in each of the three symmetry-planes of the family, and that one and only one pair of focal lines is real; which establishes the required result.

The modifications necessary in the case of an orthorhombic or monoclinic crystal are obvious.

§ 3. We proceed now to the graphical construction of the position of the optic axes. We shall suppose the sphere gnomonically projected from $O$ on to the tangent plane at any point $V$ of the sphere, and that the projections of the crystal-poles and of the extinction-directions are given. We denote the projection of a point on the sphere by the same letter as that used for the sphere ($A$ is the projection of $A$, $aa'$ of $aa'$, etc.). The problem is: Given the projections of $A, B, \ldots$ and of the extinction-directions $aa'$, $bb'$, $\ldots$; to construct the points $H$ and $K$.

Since on the sphere the planes of $aa'$ are perpendicular, on the plane the lines $aa'$ are conjugate with respect to the circle with centre $V$ and radius $\sqrt{(-1)}$. Therefore the perpendicular from $V$ on $a$ meets $a$ and $a'$ on opposite sides of $V$, and $V$ lies in the acute angle between $a$ and $a'$; and so for $B$, $C$, . . . .

We shall require the following well-known constructions:

(1) Construct the fourth ray of a harmonic pencil when three rays are given. (2) Construct the double rays of an involution pencil given by two pairs of rays (or the double points of an involution range), and the perpendicular pair of rays. (3) Construct the common pair of two involutions with the same vertex, when two pairs of rays in each involution are given. (4) Construct the self-corresponding points of two homographic ranges on a line, when three pairs of corresponding points are given.

The first construction requires ruler only; for the rest ruler-and-compass will be required in general.

### Orthorhombic Crystal.

§ 4. Take $OV$ as an axis of symmetry of the crystal, and let the other symmetry-axes of the crystal through $O$ meet the plane of projection in $U$ and $W$ (at infinity). Let $a$ and $a'$ meet $VW$, $WU$, $UV$ in $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\beta\beta'$ being at infinity. A figure shows at once that, since $V$ is in the acute angle between $a$ and $a'$, two of the three involutions ($\alpha\alpha'$, $VW$), ($\beta\beta'$, $WU$), ($\gamma\gamma'$, $UV$) are overlapping and one non-overlapping. Then $H$ and $K$ are the double points of the non-overlapping involution, and can be constructed with ruler-and-compass.

Hence one extinction-direction suffices for the unique determination of the optic axes.

### Monoclinic Crystal.

§ 5. Take $OV$ as the axis of symmetry of the crystal. By the symmetry $H$ and $K$ must be at infinity, or else $V$ must bisect $HK$.

Let $p$ and $p'$ be the lines through $V$ parallel to $a$ and $a'$, and let $q$ and $q'$ be the lines through $V$ parallel to $b$ and $b'$. Since $V$ lies in the acute angle between $aa'$, $A$ lies in the acute angle between $pp'$; and so for $B$. There are two cases to consider.

(1) If $H$ and $K$ are at infinity, $VH$ and $VK$ are evidently the double rays of the involution determined by $pp'$ and $qq'$; and $H$ and $K$ are constructed by ruler-and-compass.

(2) If $V$ bisects $HK$, $HK$ is a common diameter of the hyperbola through $A$ with $p$ and $p'$ as asymptotes and of the hyperbola through $B$

with $q$ and $q'$ as asymptotes. For two lines through $A$, which are harmonic conjugates with respect to $a$ and $a'$, are parallel to conjugate diameters of the first hyperbola, and therefore meet this hyperbola again at the ends of a diameter; and so for the second hyperbola.

Now it is readily proved by analysis that the two common diameters of any two concentric conics belong to the involution pencil determined by the two pairs of asymptotes; also that in this case the common diameters are perpendicular, using the fact that $a$ and $a'$, $b$ and $b'$ are conjugate with respect to a circle with centre $V$. The line $HK$ is therefore constructed as one of a pair of perpendicular rays belonging to the involution determined by $p$ and $p'$, $q$ and $q'$. An intersection $H$ of either of these perpendicular rays with the first hyperbola is constructed by use of the fact that the product of the perpendiculars from $H$ on $p$ and $p'$ is the same as the product of the perpendiculars from $A$.[1]

Now suppose the involution determined by $pp'$ and $qq'$ to be non-overlapping. Then in case (1) $H$ and $K$ are real. In case (2) $H$ and $K$ are unreal. For it is immediately obvious from a figure that, if two concentric hyperbolas, whose asymptotes form a non-overlapping involution and which lie each in the acute angle between its asymptotes, meet in real points at all, their common diameters cannot be perpendicular.

Now suppose the involution determined by $pp'$ and $qq'$ to be overlapping. In case (1) $H$ and $K$ are unreal. In case (2) a figure shows that two concentric hyperbolas whose asymptotes form an overlapping involution meet in two and only two real points at the ends of a diameter.

Hence the position of the optic axes of a monoclinic crystal can be determined from the extinction-directions on two faces. The solution is unique, and can be obtained by ruler-and-compass constructions.

## Triclinic Crystal.

§ 6. Suppose that the projections of four face-poles $A$, $B$, $C$, $D$ and of their extinction-directions $aa'$, $bb'$, $cc'$, $dd'$ are given. Then by § 2 the point-pair $H$, $K$ is determined as a degenerate member of the tangential

[1] The approximate positions of the two hyperbolas, and therefore of $H$ and $K$, are obvious at once when $A$, $B$, $p$, $p'$, $q$, $q'$ are drawn. It may be preferable in practice to draw the portions of each hyperbola near $H$, $K$ by freehand, and find where they meet. The first hyperbola is easily drawn by getting the point where it meets any transversal through $A$; using the fact that the parts intercepted on any line between a hyperbola and its asymptotes are equal. Similarly for the second hyperbola.

pencil of conics for all of which $aa'$, $bb'$, $cc'$, $dd'$ are conjugate pairs of lines; $HK$ being a side of the common self-conjugate triangle of the conics. There are three such pairs $HK$ (only one being real), and therefore their determination involves the solution of a cubic equation. Hence a ruler-and-compass construction for $H$, $K$ is out of the question, and it remains to consider what other comparatively simple constructions are possible.

The simplest solution theoretically will be obtained, if we assume that not only are ruler-and-compass available, but that it is possible to draw the real common tangents (or points) of two conics for each of which five real points or tangents are given.

The conics for which $aa'$, $bb'$, $cc'$, $dd'$ are conjugate have not *real* common tangents, so we cannot obtain their real common self-conjugate triangle by finding the intersections of these common tangents. But since there is a known ruler-and-compass construction for finding any number of tangents to a conic with five given pairs of conjugate lines, we may assume that five conics of the tangential pencil are obtainable (i.e. five tangents or points of each constructed). Now the polars of any fixed point with respect to the five conics are obtainable by ruler-and-compass. The conic touching these five lines touches the sides of the common self-conjugate triangle of the tangential pencil. Taking two positions of the fixed point, we get two real conics touching the sides of the common self-conjugate triangle.

§ 7. The above graphical method is solely of theoretical interest. The following method is far more practical, and is also interesting, though it is not theoretically so simple as that of § 6, since it involves the drawing of two cubic curves.

The locus of a pair of points $PP'$ such that $AP$ and $AP'$ are harmonic conjugates for $a$ and $a'$, $BP$ and $BP'$ for $b$ and $b'$, $CP$ and $CP'$ for $c$ and $c'$, is a cubic curve on which $P$ and $P'$ are a 'conjugate pair of points'; namely the Jacobian of the three line-pairs $aa'$, $bb'$, $cc'$.[1]

The cubic can be drawn with a high degree of accuracy, if a sufficient number of points on the curve is obtained.

Now in general an indefinite such number of points is obtainable with ruler only.

First of all, let the harmonic conjugate of $AB$ for $b$ and $b'$, and the harmonic conjugate of $AC$ for $c$ and $c'$ meet at $A'$. Then $A$ and $A'$ are

[1] See Hilton's 'Plane Algebraic Curves,' 1920, ch. xv. The method of constructing the cubic will be intelligible without a knowledge of the theory of cubic curves, if the proofs of some of the statements are taken for granted.

a conjugate pair on the cubic, the tangent at $A$ being the harmonic conjugate of $AA'$ for $a$ and $a'$. Similarly for $B$ and $C$.

Next let the harmonic conjugate of $BC$ for $b$ and $b'$, and the harmonic conjugate of $BC$ for $c$ and $c'$ meet at $A_1$, and let the harmonic conjugate of $AA_1$ for $a$ and $a'$ meet $BC$ at $A_1'$. Then $A_1$ and $A_1'$ are a conjugate pair on the cubic. Similarly for $B$ and $C$.

We have thus six pairs of conjugate points $AA'$, $BB'$, $CC'$, $A_1A_1'$, $B_1B_1'$, $C_1C_1'$, and the tangents at $A$, $B$, $C$. Since nine points determine a cubic and we have already fifteen (counting a tangent and its point of contact as two points), it will probably be easy to trace the cubic even now. However, more points are at once found using the fact that, if $PP'$, $QQ'$ are any two conjugate pairs of points on the cubic, so are the intersections of $PQ$ and $P'Q'$, $PQ'$ and $P'Q$. Taking any two pairs of conjugate points once found as $PP'$, $QQ'$, we get in general an infinite number of conjugate pairs of points on the cubic by ruler only.

Again, take any fixed pair of lines $pp'$ through $A$ harmonically conjugate with respect to $aa'$. Take any point $E$ on $p'$ and let the harmonic conjugates of $EB$ for $bb'$ and of $EC$ for $cc'$ meet $p$ in $S$ and $S_1$ respectively. Then as $E$ varies, $S$ and $S_1$ trace out two homographic ranges on $p$; and constructing their self-corresponding points we get the remaining intersections of $p$ with the cubic. Their construction requires ruler-and-compass, or ruler only, if $p$ is the line joining $A$ to any point of the cubic already found.

Construct a cubic similarly starting with $A$, $B$, $D$ instead of $A$, $B$, $C$. The two cubics meet in $A$, $B$, $C_1$ and in three pairs of conjugate points. The lines joining each pair are the sides of the common self-conjugate triangle of § 6. Only one pair is real, and this pair is the pair of points $HK$, which was required.

§ 8. Suppose finally that we have found the positions of $HK$, that $OV$ bisects the angle $HOK$ whose magnitude is $2\alpha$, and that we have projected gnomonically on to the tangent plane at $V$. If we now replace the gnomonic projection by its orthogonal projection on the plane through the perpendicular bisector of $HK$ making an angle $\alpha$ with the plane of the original gnomonic projection, the angle between any two lines through $H$ (or $K$) is equal to that between the original arcs on the sphere.[1] The conics for which $aa'$, $bb'$, $cc'$, $dd'$ are all conjugate become now conics with $H$ and $K$ as real foci, and $aa'$ are the bisectors of the angles between $HA$ and $KA$, as on the sphere.

[1] We leave the simple geometric proof to the reader.

This method of mapping the sphere seems very convenient for exhibiting the optical properties of the crystal. It is also very convenient when the constants of the crystal have been obtained by measurements on the two-circle goniometer of the angles between the zones which pass through two face-poles $H$, $K$ (not the optic axes) and the angles between the faces in the zone $HK$, as suggested by Fedorov. This method of mapping has also been suggested by Klingatsch [1] as convenient for the celestial sphere in astronomy; $H$ and $K$ being, for instance, the zenith and celestial north pole.

[1] A. Klingatsch, Sitzungsber. Akad. Wiss. Wien, Math.-naturw. Kl., 1914, vol. 123, Abt. II a, p. 745.