	Observed.	Calculated.
	° ′	° ′
$n\phi$	11 2	10 59
$n_{,}\phi$	33 $33\frac{1}{4}$	33 19
$\phi\phi_{,}$	22 $33\frac{1}{2}$	22 31
$c\phi$	27 $42\frac{1}{2}$	27 $47\frac{1}{3}$

Gold.

Measurement of a large though imperfect crystal in the British Museum showed it to be a combination of the cube with the tetrakishexahedron {4 1 0} and the triakisoctahedron {8 1 1}. As the faces were very dull, and but rough measurements could be obtained, I was glad to confirm this observation by the examination of a crystal showing the same combination in Mr. Ludlam's beautiful collection, which he was good enough to lend me. The angles measured on the latter crystal agree fairly well with the calculated angles.

	Measured.	Calculated.
	° ′	° ′
(4 1 0, 8 1 1)	9 25	9 52
(4 1 0, 4 0 1)	20 18	19 44

I saw recently a very beautiful crystal of the same combination in the collection of the École des Mines, Paris. In the two former the planes {8 1 1} are deeply striated parallel to their intersection with the faces of the cube.

IX. *On the Conditions of Perpendicularity in a Parallelepipedal System.* By H. J. S. Smith, *F.R.S., Savilian Professor of Geometry in the University of Oxford**.

1. The conception of a parallelepipedal system (*i. e.* of a space divided by three systems of equidistant parallel planes into similar and equal parallelepipeds) may be regarded as forming the basis of the usually received theory of crystallography. It is the object of the present note to state some of the conditions for the perpendicularity of lines and planes in such a system. The results of this inquiry (which has been undertaken at the request of Professor N. S. Maskelyne, and owes much to his suggestions) are submitted to the Crystallological Society with great diffidence, because they do not

* Read June 14, 1876.

seem likely to admit of any direct application to the practical work of the crystallographer. Such interest as they possess belongs to a domain which borders on the one hand on pure arithmetic, and on the other hand on pure geometry.

2. It is perhaps hardly necessary to explain that by a "line of the system" we understand a line joining any two points of the given parallelepipedal system, by "a plane of the system" a plane containing three points of the system, the points of the system being the points of intersection of the three sets of equidistant parallel planes by which the system is defined. It will be sufficient to consider *origin*-lines and planes, *i. e.* lines and planes passing through a fixed point of the system taken as origin.

3. Whenever a line of the system is perpendicular to a plane of the system, the system has a certain "symmetry of aspect" with regard to that plane. Let Ω be the plane, and let O be any point of the system lying in it. The planes and lines of the system which pass through O are symmetrically distributed with regard to Ω; but the points of the system are not (in general) symmetrically distributed with regard to Ω: thus, if OP is any line of the system not lying in the plane Ω, and if OQ is the *reflection* of OP with regard to the plane Ω, OQ is a line of the system as well as OP, but the points of the system which lie on OQ are not (in general) the reflections of the points of the system which lie on OP. Hence, while the points of the system are not themselves symmetrically distributed with regard to Ω, the directions in which they would be viewed by an eye situated at O are symmetrically distributed; and this is what we intend to express by saying that the system has a "symmetry of aspect" with regard to the plane Ω.

As we shall have no occasion in what follows to consider planes of absolute symmetry, we shall for the sake of brevity use the word symmetry in the sense of "symmetry of aspect." Thus any line and any plane of the system which are at right angles to one another are an axis and a plane of symmetry.

4. The cases of symmetry, as thus defined, which can present themselves in a parallelepipedal system are four in number. There is (1) the case of *simple* symmetry, when there is only one axis and one plane of symmetry; and there are three cases of triple symmetry, which may be characterized as (2) the *ellipsoidal*, (3) the spheroidal, and (4) the spherical. In an

ellipsoidal system there are three mutually rectangular planes, which are planes of symmetry; in a spheroidal system there is one equatorial plane of symmetry, but every plane of the system at right angles to this plane is also a plane of symmetry; in a system having spherical symmetry every plane of the system is a plane of symmetry, and every line of the system an axis of symmetry. Two simple symmetries cannot coexist without forming a triple symmetry, which is ellipsoidal if the axis of one of the symmetries lies in the plane of the other, but is spheroidal in every other case: three simple symmetries form an ellipsoidal symmetry if the three axes are at right angles to one another, a spheroidal symmetry if one of the axes is at right angles to the plane of the other two which are not at right angles to one another, a spherical symmetry in every other case.

5. Adopting the notation of the classical treatise of Professor W. H. Miller, we designate by a, b, c the parameters appertaining to the three lines of the system taken for the coordinate axes; we also denote by X, Y, Z the angles between the coordinate axes, and by X_1, Y_1, Z_1 the angles between the normals to the coordinate planes. We thus have for the square of the distance between any two points of the system the expression

$$f(x,y,z)=a^2x^2+b^2y^2+c^2z^2+2bcyz\cos X+2cazx\cos Y$$
$$+2abxy\cos Z,$$

where x, y, z denote any positive or negative integral numbers; and this ternary quadratic form may be regarded as characterizing the given parallelepipedal system. Again, if

$$\phi(\xi,\eta,\zeta)=b^2c^2\xi^2\sin^2 X+c^2a^2\eta^2\sin^2 Y+a^2b^2\zeta^2\sin^2 Z$$
$$+2a^2bc\eta\zeta\sin Y\sin Z\cos X_1+2b^2ca\zeta\xi\sin Z\sin X\cos Y_1$$
$$+2c^2ab\xi\eta\sin X\sin Y\cos Z_1,$$

the form ϕ, which is the contravariant of f, characterizes (in the same way in which f characterizes the given system) a new parallelepipedal system (the polar system of Auguste Bravais) in which every line is perpendicular to a plane of the given system, and in which the parameter corresponding to any line is the elementary parallelogram of the given system lying in the plane to which the line is perpendicular.

6. We write for brevity

$$f = Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'xz + 2C'xy,$$

$$\phi = A_1\xi^2 + B_1\eta^2 + C_1\zeta^2 + 2A'_1\eta\zeta + 2B'_1\xi\zeta + 2C'\xi\eta$$

(so that $A = a^2, \ldots A' = bc \cos X, \ldots, A_1 = b^2c^2 \sin^2 X, \ldots A'_1 = a^2bc \sin Y \sin Z \cos X_1, \ldots$); and we observe that although the five quantities upon which the nature of the parallelepipedal system ultimately depends are the ratios of the parameters a, b, c, and the three angles X, Y, Z, yet the combinations of these quantities which it is most convenient to consider in discussing the conditions of perpendicularity are precisely the six coefficients

$$A, B, C, \quad A', B', C',$$

and the six contravariant coefficients

$$A_1, B_1, C_1, \quad A'_1, B'_1, C'_1.$$

Thus the condition that the lines of the system

$$\left.\begin{array}{l} \dfrac{x}{au} = \dfrac{y}{bv} = \dfrac{3}{cw}, \\ \dfrac{x}{au_1} = \dfrac{y}{bv_1} = \dfrac{3}{cw_1}, \end{array}\right\} \quad \ldots \ldots \quad \text{(i)}$$

should be perpendicular to one another is

$$u_1 \frac{df}{du} + v_1 \frac{df}{dv} + w_1 \frac{df}{dw} = 0,$$

or

$$u \frac{df}{du_1} + v \frac{df}{dv_1} + w \frac{df}{dw_1} = 0;$$

the condition that the planes of the system

$$\left.\begin{array}{l} \dfrac{hx}{a} + \dfrac{ly}{b} + \dfrac{kz}{c} = 0, \\ \dfrac{h_1x}{a} + \dfrac{l_1y}{b} + \dfrac{k_1z}{c} = 0 \end{array}\right\} \quad \ldots \ldots \quad \text{(ii)}$$

should be perpendicular to one another is

$$h_1 \frac{d\phi}{dh} + k_1 \frac{d\phi}{dk} + l_1 \frac{d\phi}{dl} = 0,$$

or

$$h \frac{d\phi}{dh_1} + k \frac{d\phi}{dk_1} + l \frac{d\phi}{dl_1} = 0;$$

and the conditions that the first of the lines (i) should be perpendicular to the first of the planes (ii) may be written in one or other of the equivalent forms

$$\frac{\left(\frac{df}{du}\right)}{h}=\frac{\left(\frac{df}{dv}\right)}{k}=\frac{\left(\frac{df}{dw}\right)}{l},$$

$$\frac{\left(\frac{d\phi}{dh}\right)}{u}=\frac{\left(\frac{d\phi}{dk}\right)}{v}=\frac{\left(\frac{d\phi}{dl}\right)}{w}.$$

7. Let us now suppose that the given parallelepipedal system contains a pair of perpendicular lines (i); the condition of perpendicularity gives immediately

$$\begin{aligned}\mathrm{A}uu_1+\mathrm{B}vv_1+\mathrm{C}ww_1+\mathrm{A}'(vw_1+wv_1)+\mathrm{B}'(wu_1+w_1u)\\+\mathrm{C}'(uv_1+u_1v)=0.\end{aligned}$$

Unless, therefore, the six covariant coefficients are connected by a linear homogeneous relation having integral coefficients, no two lines of the system can be perpendicular to one another; and correlatively, unless the six contravariant coefficients are connected by a similar relation, no two planes of the system can be perpendicular to one another. But the existence of such a relation connecting the six covariant coefficients (or the six contravariant coefficients), though a necessary condition, is not a sufficient condition for the existence of a pair of perpendicular lines or planes. We proceed, therefore, very briefly to describe the principal cases which present themselves when the coefficients are connected by one, two, three, four, or five linear relations. By a linear relation connecting the coefficients we understand a linear homogeneous equation of the type

$$p\mathrm{A}+q\mathrm{B}+r\mathrm{C}+2p'\mathrm{A}'+2q'\mathrm{B}'+2r'\mathrm{C}'=0,$$

where p, q, r, p', q', r' are integral numbers which we may suppose free from any common divisor. In connexion with such a relation we shall have to consider the quadratic form

$$\psi=p\xi^2+q\eta^2+r\zeta^2+2p'\eta\zeta+2q'\zeta\xi+2r'\xi\eta$$

and its contravariant or reciprocal form

$$\begin{aligned}\Psi=(p'^2-qr)x^2+(q'^2-rp)y^2+(r'-pq)z^2+2(pp'-q'r')yz\\+2(qq'-r'p')zx+2(rr'-p'q')xy.\end{aligned}$$

These we shall term the quadratic form and the reciprocal quadratic form appertaining to the given relation. For brevity we shall attend only to the cases in which given relations exist between the six covariant coefficients A, B, C, A′, B′, C′, the cases in which given relations exist between the six contravariant conditions being simply the correlatives of these. It is remarkable that in every case the conditions of perpendicularity and symmetry depend solely on the coefficients of the linear relations connecting the crystallographic coefficients; so that two parallelepipedal systems, in which the crystallographic coefficients have different ratios but satisfy the same linear relations, would resemble one another exactly in respect of symmetry and perpendicularity.

8. *Case of one linear relation between the coefficients.*

Here we have the theorem, "The system contains a single pair of perpendicular lines, or contains no such pair whatever, according as the reciprocal form appertaining to the given relation is or is not a perfect square."

For the condition that the reciprocal form Ψ should be a perfect square, we may if we please substitute the condition that the quadratic form ψ appertaining to the given relation should resolve itself into two rational factors. Or, again, we may replace this condition by the two conditions, (1) that the discriminant of ψ is to be zero, (2) that the greatest common divisor of the first minors of this discriminant is to be a perfect square.

9. *Case of two linear relations between the coefficients.*

We represent the quadratic forms and the reciprocal quadratic forms appertaining to these relations by $\psi_1\Psi_1$, $\psi_2\Psi_2$, and by $\theta, \theta', \theta''$ the roots of the discriminantal cubic of $\psi_1+\theta\psi_2$. If these roots are irrational, the system contains not a single pair of perpendicular lines. If one of them, for example θ, is rational, we still have to examine whether the factors of $\psi_1+\theta\psi_2$ are rational; if they are, we have a pair of perpendicular lines. If all the three roots $\theta, \theta', \theta''$ are rational, we have to examine the factors of each of the three forms $\psi_1+\theta\psi_2$, $\psi_1+\theta'\psi_2$, $\psi_1+\theta''\psi_2$; according as these factors are or are not rational (if the factors of two of them are rational the factors of the third are so too), we obtain one or three pairs of perpendicular lines, or no pair at all of such lines.

When two of the roots $\theta, \theta', \theta''$ are equal, we have either one, and only one, pair of perpendicular lines; or we may have two pairs, the plane of one of the right angles containing one of the rays of the other right angle. When the three roots are all equal we have a single pair of perpendicular lines.

Lastly, the coefficients of the discriminating cubic may all vanish. If this happens, either (α) Ψ_1 and Ψ_2 differ, if at all, by a numerical factor, and every line of the system that lies in a certain plane has a line of the system at right angles to it in the same plane; or (β) ψ_1 and ψ_2 have a common linear factor, and the system possesses a simple symmetry.

We may thus enunciate the theorem:—

"The conditions that a parallelepipedal system should possess a simple symmetry are (a) that the coefficients should be connected by two linear relations, (b) that the two quadratic forms appertaining to these relations should have a linear factor in common."

10. *Case of three linear relations between the coefficients.*

We represent by ψ_1, ψ_2, ψ_3 the quadratic forms appertaining to the given relations, and we obtain the following theorem:—

"The system contains no right angle, or an infinite number, according as the indeterminate cubic equation

$$C = \begin{vmatrix} \frac{d\psi_1}{d\xi}, & \frac{d\psi_1}{d\eta}, & \frac{d\psi_1}{d\zeta} \\ \frac{d\psi_2}{d\xi}, & \frac{d\psi_2}{d\eta}, & \frac{d\psi_2}{d\zeta} \\ \frac{d\psi_3}{d\xi}, & \frac{d\psi_3}{d\eta}, & \frac{d\psi_3}{d\zeta} \end{vmatrix} = 0$$

does not or does admit of solution in integral numbers."

By virtue of the three given relations the characteristic expression $f(x, y, z)$ of art. 5 assumes the form

$$f(x, y, z) = \omega_1 f_1 + \omega_2 f_2 + \omega_3 f_3,$$

the ratios of the quantities $\omega_1, \omega_2, \omega_3$ being irrational, but the coefficients of the quadratic forms f_1, f_2, f_3 being integral numbers. If $H(x, y, z)$ denote the Jacobian of these three forms, we have the theorem:—

"When the indeterminate equation $C=0$ admits of solution, the infinite number of right angles which the system contains all lie on the cubic cone $H(xa, yb, zc)=0$; viz. an infinite number of lines of the system lie on this cone, and every line of the system which lies on it has a line at right angles to it, also lying on the cone."

The system may have a simple symmetry or an ellipsoidal symmetry, or none at all; but it cannot have a spheroidal or a spherical symmetry.

The conditions for a simple symmetry are that the ternary cubic form $C(\xi, \eta, \zeta)$ should resolve itself into a rational linear factor and a rational quadratic factor, and that the ternary cubic form $H(x,y,z)$ should resolve itself into three linear factors. These conditions admit of being further developed (see Dr. Salmon's 'Higher Plane Curves,' pp. 190 and 202 *seqq.*); it is sufficient for our purpose to observe that the coefficients of the Jacobian $H(x, y, z)$, no less than those of $C(\xi, \eta, \zeta)$, depend solely on the coefficients of the forms ψ_1, ψ_2, ψ_3, *i. e.* on the integral numbers entering into the given linear relations.

The conditions for an ellipsoidal symmetry are that $C(\xi,\eta,\zeta)$ should resolve itself into three rational linear factors, and that $H(x,y,z)$ should resolve itself into three factors.

Two special cases of the general theory (which, however, are not cases of symmetry) deserve attention.

(1) There may exist in the parallelepipedal system a quadratic cone and a plane, such that every line of the system lying in the plane has a line of the system at right angles to it lying in the cone.

(2) Or, again, the parallelepipedal system may have an infinite number of pairs of perpendicular lines all lying in the same plane; and it may also have at the same time a second set of such pairs lying on the surface of a quadratic cone, the plane of each pair of this second set passing through the polar line of the first-named pair with regard to the cone.

11. *Case of four linear relations between the coefficients.*

Here every line, without exception, of the parallelepipedal system has a line at right angles to it; and this distribution of pairs of perpendicular lines may exist without the presence of any symmetry whatever. The symmetry (if any) may be simple, or ellipsoidal, or spheroidal, but cannot be spherical.

The characteristic form $f(x, y, z)$ may be expressed by an equation of the type

$$f = \omega_1 f_1 + \omega_2 f_2,$$

the ratio of ω_1 and ω_2 being irrational, but the coefficients of the quadratic forms f_1 and f_2 being integral numbers. There is a simple symmetry when the discriminantal cubic of $f_1 + \theta f_2$ has one rational root, an ellipsoidal symmetry when it has three rational and unequal roots, a spheroidal symmetry when it has two equal roots. (It cannot have its three roots equal, because the cone $f(x, y, z) = 0$ is imaginary.)

We suppress the further discussion of these conditions, only observing that they may be so expressed as to show that they depend only on the coefficients of the four given relations, and not on the six coefficients A, B, C, A′, B′, C′ themselves.

12. *Case of five linear relations between the coefficients.*

In this case the ratios of the coefficients are themselves evidently rational, and the parallelepipedal system has a spherical symmetry. It is also true, conversely, that when there is a spherical symmetry the ratios of the coefficients are rational.

We may mention that the question of the rationality or irrationality of the ratios of the crystallographic coefficients had attracted the attention of Gauss, who, as appears from the memoir of his life (Gauss, *Zum Gedächtniss*, von W. Sartorius v. Waltershausen: Leipzig, 1856), had in the year 1831 devoted himself with great ardour to the study of crystallography *.

* Some of the demonstrations, which have been omitted in the present note, will be found in a paper inserted in the 'Proceedings of the London Mathematical Society,' vol. vii. p. 83.